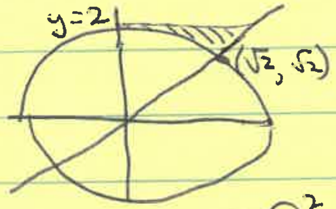
MATH 213 FALL 2025 EXAM

SOLUTIONS

1. $x = \sqrt{4-y^2} \Rightarrow x^2 = 4-y^2 \Rightarrow x^2+y^2=4$ circle of radius 2 around the origin

$x = y$ line, slope 1, through origin

Intersection when $x=y$ & $x^2+y^2=4$



$\Rightarrow \quad 2x^2 = 4 \Rightarrow x^2 = 2, \quad x = \pm\sqrt{2}, y = \pm\sqrt{2}$

y goes from $\sqrt{2}$ to 2

Note $y=2 \Rightarrow r\sin\theta = 2 \Rightarrow r = \frac{2}{\sin\theta}$

$$\Rightarrow \int_{\sqrt{2}}^{2}\int_{\sqrt{4-y^2}}^{y} dx\,dy = \underline{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{2}^{\frac{2}{\sin\theta}} r\,dr\,d\theta}$$

The region (polar) goes from the circle out to the line $y=2$ or $r = {}^{2}/_{\sin\theta} = 2\csc\theta$

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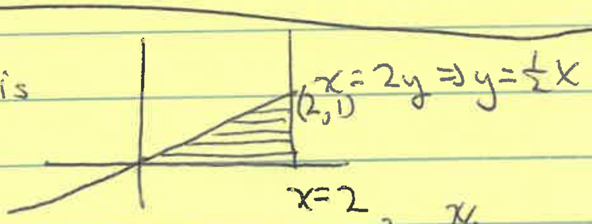
2. The region in x-y is

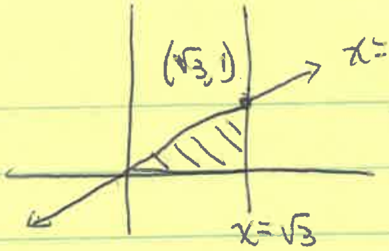


$$\Rightarrow \int_0^9\int_0^1\int_{2y}^{2} \frac{2\cos(x^2)}{\sqrt{z}}\,dx\,dy\,dz = \int_0^9\int_0^2\int_0^{x/2} \frac{2\cos(x^2)}{\sqrt{z}}\,dy\,dx\,dz$$

$$= \int_0^9\int_0^2 \frac{2\cos(x^2)y}{\sqrt{z}}\Big|_0^{x/2} dx\,dz = \int_0^9\int_0^2 \frac{x\cos(x^2)}{\sqrt{z}}\,dx\,dz$$

$$= \int_0^9 \frac{\sin(x^2)}{2\sqrt{z}}\Big|_0^2 dz = \int_0^9 \frac{\sin 4}{2\sqrt{z}}\,dz = \frac{\sin 4}{\cancel{2}}\,\cancel{2}z^{\frac{1}{2}}\Big|_0^9 = \underline{3\sin 4}$$

3.



$$\text{Volume}=\int_0^{\pi/6}\int_0^{\sqrt{3}/\cos\theta}\int_0^{6-r^2\cos^2\theta} dz\ r\,dr\,d\theta$$

note: $x=\sqrt{3} \Rightarrow r\cos\theta=\sqrt{3} \Rightarrow r=\sqrt{3}/\cos\theta$

angle is $\pi/6$ radians

$6-x^2= 6-r^2\cos^2\theta$

4.

radius is 4

ϕ goes from 0 to $\pi/6$

θ goes from 0 to 2π

$$\Rightarrow \text{Volume} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^4 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

5. $x=\frac{u}{v}$ $y=uv$ $\Rightarrow xy=u^2$ \$ $\frac{y}{x}=v^2$

$xy=1 \Rightarrow u^2=1$ $u=1$

$xy=9 \Rightarrow u^2=9$ $u=3$

$y=x \Rightarrow uv=\frac{u}{v} \Rightarrow v^2=1$, $v=1$

$y=4x \Rightarrow uv=4\frac{u}{v} \Rightarrow v^2=4$, $v=2$

$$\Rightarrow \iint\left(\sqrt{\tfrac{y}{x}}+\sqrt{xy}\right)dx\,dy = \iint(v+u)\,J\,du\,dv$$

$$J=\begin{vmatrix} x_u & x_v \\ y_u & y_v\end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u\end{vmatrix} = \frac{u}{v}+\frac{u}{v}=\frac{2u}{v}$$

$$\Rightarrow \int_1^2\int_1^3 (v+u)\frac{2u}{v}\,du\,dv = \int_1^2\int_1^3\left(2u+\frac{2u^2}{v}\right)du\,dv$$

$$= \int_1^2\left(u^2+\frac{2u^3}{3v}\Big|_1^3\right)dv = \int_1^2\left(9+\frac{18}{v}-1-\frac{2}{3v}\right)dv = \int_1^2\left(8+\frac{52}{3v}\right)dv$$

$$= \left(8v+\frac{52}{3}\ln v\right)\Big|_1^2 = 16+\frac{52}{3}\ln 2-8$$

$$= 8+\frac{52}{3}\ln 2$$

6. $\int (2x - y + z)\, ds$

let $x = 2$ $\quad \vec{r} = 2i + (1-t)\vec{j} + t\vec{k}$

$y = 1 - t \quad \Rightarrow \vec{v} = 0i - \vec{j} + \vec{k}$

$z = t \quad |\vec{v}| = \sqrt{1^2+1^2} = \sqrt{2}$

$0 \le t \le 1$

$\Rightarrow \int_0^1 (4 - 1 + t + t)\sqrt{2}\, dt = \int_0^1 (3+2t)\sqrt{2}\, dt = (3t + t^2)\sqrt{2}\Big|_0^1$

$= (3+1)\sqrt{2} - 0 = 4\sqrt{2}$

7. $\int \vec{F} \cdot \vec{n}\, ds$ circle of radius 2 has $x = 2\cos t$ $y = 2\sin t$ $0 \le t \le 2\pi$

$\int_0^{2\pi} \langle x, 2x \rangle \cdot \langle dy/dt, -dx/dt \rangle\, dt$

$\int_0^{2\pi} 2\cos t\, 2\cos t + 4\cos t\,(+2\sin t)\, dt$

$= 4\int_0^{2\pi} \cos^2 t\, dt + 8\int_0^{2\pi} \cos t \sin t\, dt$

$= 4\int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt + 8\, \frac{\sin^2 t}{2}\Big|_0^{2\pi}$

$= 2\int_0^{2\pi} dt + 2\int_0^{2\pi} \frac{\cos 2t}{2}\, dt$

$\searrow 0 \qquad \searrow 0$

$= 2 \cdot 2\pi = \underline{\underline{4\pi}}$

(Check w/ Green's Theorem

$\iint (M_x + N_y)\, dA = \underset{\checkmark}{\iint (1+0)\, dA} = 1 \cdot \pi\, 2^2 = 4\pi$)

8. $M = e^x \sin y$ $\quad$ Does $M_y = N_x$?

$N = e^x \cos y + 2y$ $\quad e^x \cos y = e^x \cos y$ ✓ $\Rightarrow \vec{F}$ is a gradient field

$f = \int e^x \sin y\, dx = e^x \sin y + g(y)$

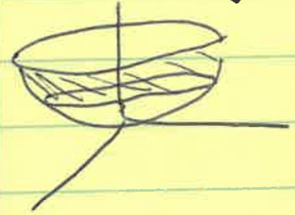
$= \int (e^x \cos y + 2y)\, dy = e^x \sin y + y^2 + h(x)$

$\Rightarrow \underline{f = e^x \sin y + y^2 + C}$

9. $\oint \vec{F} \cdot \vec{T}\, ds = \iint (N_x - M_y)\, dA dy$

$N_x = 2 \quad M_y = 0$ $\Rightarrow$ circulation $= 2 \int dA = 2\pi 1^2 = 2\pi$

10. $z = x^2 + y^2$



Surface area $= \iint \sqrt{1 + f_x^2 + f_y^2}\; r\, dr\, d\theta$

$f = x^2 + y^2 - z$

$x^2 + y^2 = 4 \Rightarrow r = 2$

$x^2 + y^2 = 9 \Rightarrow r = 3$

$f_x = 2x \,;\, f_y = 2y$

$r^2 = x^2 + y^2$

$= \iint \sqrt{1+4x^2+4y^2}\; r dr d\theta$

$= \int_0^{2\pi}\int_2^3 \sqrt{1+4r^2}\; r dr d\theta$

$= \int_0^{2\pi} \frac{2}{8 \cdot 3}(1+4r^2)^{3/2} \Big|_2^3 d\theta$

$= \left[\frac{2}{3 \cdot 8}(1+36)^{3/2} - \frac{2}{8 \cdot 3}(1+16)^{3/2}\right] 2\pi$

$= \frac{4\pi}{3 \cdot 8}\left[37^{3/2} - 17^{3/2}\right]$

$= \frac{\pi}{6}\left[37^{3/2} - 17^{3/2}\right]$